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# SMOOTHNESS IN THE $L_p$ MINKOWSKI PROBLEM FOR $p < 1$

GABRIELE BIANCHI, KÁROLY J. BÖRÖCZKY, AND ANDREA COLESANTI

ABSTRACT. We discuss the smoothness and strict convexity of the solution of the  $L_p$ -Minkowski problem when  $p < 1$  and the given measure has a positive density function.

## 1. INTRODUCTION

Given  $K$  in the class  $\mathcal{K}_0^n$  of compact convex sets in  $\mathbb{R}^n$  that have non-empty interior and contain the origin  $o$ , we write  $h_K$  and  $S_K$  to denote its support function and its surface area measure, respectively, and for  $p \in \mathbb{R}$ ,  $S_{K,p}$  to denote its  $L_p$ -area measure, where  $dS_{K,p} = h_K^{1-p} dS_K$ . The  $L_p$ -area measure defined by Lutwak [35] is a central notion in convexity, see say Barthe, Guédon, Mendelson and Naor [2], Böröczky, Lutwak, Yang and Zhang [5], Campi and Gronchi [10], Chou [15], Cianchi, Lutwak, Yang and Zhang [17], Gage and Hamilton [19], Haberl and Parapatits [23], Haberl and Schuster [24, 25], Haberl, Schuster and Xiao [26], He, Leng and Li [27], Henk and Linke [28], Ludwig [34], Lutwak, Yang and Zhang [37, 38], Naor [41], Naor and Romik [42], Paouris [44], Paouris and Werner [45] and Stancu [50].

The  $L_p$  Minkowski problem asks for the existence of a convex body  $K \in \mathcal{K}_0^n$  whose  $L_p$  area measure is a given finite Borel measure  $\nu$  on  $S^{n-1}$ . When  $p = 1$  this is the classical Minkowski problem solved by Minkowski [40] for polytopes, and by Alexandrov [1] and Fenchel and Jessen [18] in general. The smoothness of the solution was clarified in a series of papers by Nirenberg [43], Cheng and Yau [14], Pogorelov [46] and Caffarelli [7, 8]. For  $p > 1$  and  $p \neq n$ , the  $L_p$  Minkowski problem has a unique solution according to Chou and Wang [16], Guan and Lin [22] and Hug, Lutwak, Yang and Zhang [30]. The smoothness of the solution is discussed in Chou and Wang [16], Huang and Lu [29] and Lutwak and Oliker [36]. In addition, the case  $p < 1$  has been intensively investigated by Böröczky, Lutwak, Yang and Zhang [4], Böröczky and Hai T. Trinh [6], Chen [13], Chen, Li and Zhu [11, 12], Ivaki [31], Jiang [32], Lu and Wang [33], Lutwak, Yang and Zhang [39], Stancu [48, 49] and Zhu [52–55].

The solution of the  $L_p$ -Minkowski problem may not be unique for  $p < 1$  according to Chen, Li and Zhu [12] if  $0 < p < 1$ , according to Stancu [49] if  $p = 0$ , and according to Chou and Wang [16] if  $p < 0$  small.

In this paper we are interested in this problem when  $p < 1$  and  $\nu$  is a measure with density with respect to the Hausdorff measure  $\mathcal{H}^{n-1}$  on  $S^{n-1}$ , i.e. in the problem

$$(1.1) \quad dS_{K,p} = f d\mathcal{H}^{n-1} \quad \text{on } S^{n-1},$$

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where  $f$  is a non-negative Borel function in  $S^{n-1}$ .

According to Chou and Wang [16], if  $-n < p < 1$  and the Borel function  $f$  is bounded from above and below by positive constants, then (1.1) has a solution. More general existence results are provided by the recent works Chen, Li and Zhu [11] if  $p = 0$ , Chen, Li and Zhu [12] if  $0 < p < 1$ , and Bianchi, Böröczky and Colesanti [3] if  $-n < p < 0$ . In particular, it is known that (1.1) has a solution if  $0 \leq p < 1$  and  $f$  is any non-negative function in  $L_1(S^{n-1})$  with  $\int_{S^{n-1}} f d\mathcal{H}^{n-1} > 0$ , and if  $-n < p < 0$  and  $f$  is any non-negative function in  $L_{\frac{n}{n+p}}(S^{n-1})$  with  $\int_{S^{n-1}} f d\mathcal{H}^{n-1} > 0$ .

We observe that  $h$  is a non-negative positively 1-homogeneous convex function in  $\mathbb{R}^n$  which solves the Monge-Ampère equation

$$(1.2) \quad h^{1-p} \det(\nabla^2 h + hI) = f \quad \text{on } S^{n-1}$$

in the sense of measure if and only if  $h$  is the support function of a convex body  $K \in \mathcal{K}_0^n$  which is the solution of (1.1) (see Section 2). Here  $h$  is the unknown non-negative (support) function on  $S^{n-1}$  to be found,  $\nabla^2 h$  denote the (covariant) Hessian matrix of  $h$  with respect to an orthonormal frame on  $S^{n-1}$ , and  $I$  is the identity matrix. The function  $h$  may vanish somewhere even in the case when  $f$  is positive and continuous, and when this happens and  $p < 1$  the equation (1.2) is singular at the zero set of  $h$ . Naturally, if  $h$  is  $C^2$ , then (1.2) is a proper Monge-Ampère equation.

In this paper we study the smoothness and strict convexity of a solution  $K \in \mathcal{K}_0^n$  of (1.1) assuming  $\tau_2 > f > \tau_1$  for some constants  $\tau_2 > \tau_1 > 0$ . Concerning these aspects for  $p < 1$ , we summarize the known results in Theorem 1.1, and the new results in Theorem 1.2.

We say that  $x \in \partial K$  is a  $C^1$ -smooth point if there is a unique tangent hyperplane to  $K$  at  $x$ , and observe that  $\partial K$  is  $C^1$  if and only if each  $x \in \partial K$  is  $C^1$ -smooth (see Section 2 for all definitions). In addition, we note that  $h_K$  is  $C^1$  on  $S^{n-1}$  if and only if  $K$  is strictly convex, and  $h_K$  is strictly convex on any hyperplane avoiding the origin if and only if  $\partial K$  is  $C^1$ . For  $z \in \partial K$ , the exterior normal cone at  $z$  is denoted by  $N(K, z)$ , and for  $z \in \text{int } K$ , we set  $N(K, z) = \{o\}$ . Theorem 1.1 (i) and (ii) are essentially due to Caffarelli [7] (see Theorem 3.6), and Theorem 1.1 (iii) is due to Chou and Wang [16]. If the function  $f$  in (1.1) is  $C^\alpha$  for  $\alpha > 0$ , then Caffarelli [8] proves (iv).

**Theorem 1.1** (Caffarelli, Chou, Wang). *If  $K \in \mathcal{K}_0^n$  is a solution of (1.1) for  $n \geq 2$  and  $p < 1$ , and  $f$  is bounded from above and below by positive constants, then the following assertions hold:*

- (i) *The set  $X_0$  of the points  $x \in \partial K$  with  $N(K, x) \subset N(K, o)$  is closed, each point of  $X = \partial K \setminus X_0$  is  $C^1$ -smooth and  $X$  contains no segment.*
- (ii) *If  $o \in \partial K$  is a  $C^1$ -smooth point, then  $\partial K$  is  $C^1$ .*
- (iii) *If  $p \leq 2 - n$ , then  $o \in \text{int } K$ , and hence  $K$  is strictly convex and  $\partial K$  is  $C^1$ .*
- (iv) *If  $o \in \text{int } K$  and the function  $f$  in (1.1) is positive and  $C^\alpha$ , for some  $\alpha > 0$ , then  $\partial K$  is  $C^{2,\alpha}$ .*

Concerning strict convexity, Assertion (iii) here is optimal because Example 4.2 shows that if  $2 - n < p < 1$ , then it is possible that  $o$  belongs to the relative interior of an  $(n - 1)$ -dimensional face of a solution  $K$  of (1.1) where  $f$  is a positive continuous function. Therefore, the only question left open is the  $C^1$  smoothness of the boundary of the solution if  $2 - n < p < 1$ .

We note that if  $p < 1$  and  $K$  is a solution of (1.2) with  $f$  positive and  $o \in \partial K$ , then

$$(1.3) \quad \dim N(K, o) \leq n - 1.$$

Therefore, Theorem 1.1 (ii) yields that  $\partial K$  is  $C^1$  for the solution  $K$  if  $n = 2$ . In general, we have the following partial results.

**Theorem 1.2.** *If  $K \in \mathcal{K}_0^n$  is a solution of (1.1) for  $n \geq 2$  and  $p < 1$ , and  $f$  is bounded from above and below by positive constants, then the following assertions hold:*

- (i) *If  $n = 2$ ,  $n = 3$  or  $n > 3$  and  $p < 4 - n$ , then  $\partial K$  is  $C^1$ .*
- (ii) *If  $\mathcal{H}^{n-1}(X_0) = 0$  for the  $X_0$  in Theorem 1.1 (i), then  $\partial K$  is  $C^1$ .*

Our results differ in some cases from the ones in Chou and Wang [16], possibly because [16] considers the equation

$$(1.4) \quad \det(\nabla^2 h + hI) = fh^{p-1} \quad \text{on } S^{n-1}$$

instead of (1.2). In the context of non-negative convex functions, being a solution of this last equation is a priori more restrictive than being a solution of (1.2), even if obviously the two notions coincide when  $h$  is positive (see Section 2 for more on this point). Chou and Wang [16] proves, under our same assumptions on  $f$ , the strict convexity of the solution  $h$  of (1.4) on hyperplanes avoiding the origin, and uses this to prove that  $\partial K$  is  $C^1$  for the convex body  $K$ . We note that if  $K \in \mathcal{K}_0^n$  is a solution of (1.4) for  $p < 1$  and  $f$  is bounded from below and above by positive constants, then combining Theorem 1.2 (ii) with the simple observation (2.11) in Section 2 shows that  $\partial K$  is  $C^1$ , as it was verified by Chou and Wang [16]. In our opinion (1.2) is the right equation to consider and using it we obtain weaker results.

To give an example of how the two equations differ, the support function  $h$  of the body  $K$  in Example 4.2 (where  $o$  belongs to the relative interior of an  $(n-1)$ -dimensional face) is a solution of (1.2) but not a solution of (1.4).

According to Chou and Wang [16] (see also Lemma 3.1 below), the Monge-Ampère equation (1.2) can be transferred to a Monge-Ampère equation

$$(1.5) \quad v^{1-p} \det(D^2 v) = g$$

for a convex function  $v$  on  $\mathbb{R}^{n-1}$  where  $g$  is a given non-negative function and  $D^2$  stands for the Hessian in  $\mathbb{R}^{n-1}$ .

The proofs of Claims (i) and (ii) in Theorem 1.1 use as an essential tool a result proved by Caffarelli in [7] regarding smoothness and strict convexity of convex solutions of certain Monge-Ampère equation of type (1.5) (see Theorem 3.6). Proving that  $\partial K$  is  $C^1$  is equivalent to prove that  $h_K$  is strictly convex, and [7] is the key to prove this property in  $\{y \in S^{n-1} : h_K(y) > 0\}$ .

The proof of Claim (i) in Theorem 1.2 is based on the following result for the singular inequality  $v^{1-p} \det D^2 v \geq g$ .

**Proposition 1.3.** *Let  $\Omega \subset \mathbb{R}^n$  be an open convex set, and let  $v$  be a non-negative convex function in  $\Omega$  with  $S = \{x \in \Omega : v(x) = 0\}$ . If for  $p < 1$  and  $\tau > 0$ ,  $v$  is the solution of*

$$(1.6) \quad v^{1-p} \det D^2 v \geq \tau \quad \text{in } \Omega \setminus S$$

*in the sense of measure, and  $S$  is  $r$ -dimensional, for  $r \geq 1$ , then  $p \geq -n + 1 + 2r$ .*

We mention that in Caffarelli [9] a corresponding result for  $p = 1$  is established.

The underlying idea behind the proof of this result is the following: On the one hand, the graph of  $v$  near  $S$  is close to being ruled. Hence, the total variation of the derivative is “small”. On the other hand, the total variation of the derivative is “large” because of the Monge-Ampère inequality (1.6).

The inequality  $p \geq -n + 1 + 2r$  in this result is close to being optimal, at least when  $r = 1$ . Indeed, Example 3.2 shows that, for any  $p > -n + 3$ , there exists a non-negative convex solution of (1.6) in  $\Omega$  which vanishes on the intersection of

$\Omega$  with a line. For the version  $p = 1$  of Proposition 1.3, Caffarelli [9] proves that  $\dim S < n/2$  and that this inequality is optimal.

Proposition 1.3 yields actually somewhat more than Claim (i) in Theorem 1.2; namely, if  $r \geq 2$  is an integer,  $p < \min\{1, 2r - n\}$  and  $K \in \mathcal{K}_0^n$  is a solution of (1.1) with  $o \in \partial K$ , then  $\dim N(K, o) < r$ . As a consequence, we have the following technical statements about  $K$ , where we also use Theorem 1.2 (ii) for Claim (ii).

**Corollary 1.4.** *If  $p < 1$  and  $K \in \mathcal{K}_0^n$ ,  $n \geq 4$ , is a solution of (1.1) with  $o \in \partial K$ , then*

- (i)  $\dim N(K, o) < \frac{n+1}{2}$ ;
- (ii) *if in addition  $n = 4, 5$  and  $\partial K$  is not  $C^1$ , then  $\dim N(K, o) = 2$  and  $\dim F(K, u) = n - 1$  for some  $u \in N(K, o)$ .*

In Section 2 we review the notation used in this paper. Section 3 contains results and examples regarding Monge-Ampère equations in  $\mathbb{R}^n$ , namely Proposition 1.3, Example 3.2 and Proposition 3.4. This last result is the key to prove Theorem 1.2 (ii). In Section 4 we show, for the sake of completeness, how to prove Theorem 1.1 using ideas due to Caffarelli [7, 8] and Chou and Wang [16]. Theorem 1.2 and Corollary 1.4 are proved in Section 5.

## 2. NOTATION AND PRELIMINARIES

As usual,  $S^{n-1}$  denotes the unit sphere and  $o$  the origin in the Euclidean  $n$ -space  $\mathbb{R}^n$ . The symbol  $B^n$  denotes the unit ball in  $\mathbb{R}^n$  centred at  $o$  and  $\omega_n$  denotes its volume. If  $x, y \in \mathbb{R}^n$ , then  $\langle x, y \rangle$  is the scalar product of  $x$  and  $y$ , while  $\|x\|$  is the euclidean norm of  $x$ . By  $[x, y]$  we denote the segment with endpoint  $x$  and  $y$ .

We write  $\mathcal{H}^k$  for  $k$ -dimensional Hausdorff measure in  $\mathbb{R}^n$ .

We denote by  $\partial E$ ,  $\text{int} E$ ,  $\text{cl} E$ , and  $1_E$  the *boundary*, *interior*, *closure*, and *characteristic function* of a set  $E$  in  $\mathbb{R}^n$ , respectively. The symbols  $\text{aff} E$  and  $\text{lin} E$  denote respectively the *affine hull* and the *linear hull* of  $E$ . The *dimension*  $\dim E$  is the dimension of  $\text{aff} E$ . With the symbol  $E \mid L$  we denote the orthogonal projection of  $E$  on the linear space  $L$ .

Given a function  $v$  defined on a subset of  $\mathbb{R}^n$ ,  $Dv$  and  $D^2v$  denote its gradient and its Hessian, respectively.

Our next goal is to recall a standard notion of generalised solution of Monge-Ampère equations, usually referred to as *solution in the sense of measure*. Our general reference for notions and facts about Monge-Ampère equations is the survey by Trudinger and Wang [51]. Let  $v$  be a convex function defined in an open convex set  $\Omega$ ; the subgradient  $\partial v(x)$  of  $v$  at  $x \in \Omega$  is defined as

$$\partial v(x) = \{z \in \mathbb{R}^n : v(y) \geq v(x) + \langle z, y - x \rangle \text{ for each } y \in \Omega\},$$

which is a non-empty compact convex set. Note that  $v$  is differentiable at  $x \in \Omega$  if and only if  $\partial v(x)$  consists of exactly one vector, which is the gradient of  $v$  at  $x$ . If  $\omega \subset \Omega$  is a Borel set, then we denote by  $N_v(\omega)$  the image of  $\omega$  through the gradient map of  $v$ , i.e.

$$N_v(\omega) = \bigcup_{x \in \omega} \partial v(x).$$

Note that as  $\omega$  is a Borel set, then  $N_v(\omega)$  is measurable. Hence, we may define the Monge-Ampère measure associated to  $v$  as follows

$$(2.1) \quad \mu_v(\omega) = \mathcal{H}^n(N_v(\omega)).$$

For  $p < 1$  and non-negative  $g$  on  $\mathbb{R}^n$ , we say that the non-negative convex function  $v$  satisfies the Monge-Ampère equation

$$v^{1-p} \det(D^2v) = g$$

in the sense of measure (or in the Alexandrov sense) if

$$v^{1-p} d\mu_v = g d\mathcal{H}^n.$$

Equivalently

$$\int_{\omega} v^{1-p}(x) d\mu_v(x) = \int_{\omega} g(x) dx$$

for every Borel subset  $\omega$  of  $\Omega$ .

A *convex body* in  $\mathbb{R}^n$  is a compact convex set with nonempty interior. The treatises Gardner [20], Gruber [21] and Schneider [47] are excellent general references for convex geometry. The function

$$h_K(u) = \max\{\langle u, y \rangle : y \in K\},$$

for  $u \in \mathbb{R}^n$ , is the *support function* of  $K$ . When it is clear the convex body to which we refer we will drop the subscript  $K$  from  $h_K$  and write simply  $h$ . Any convex body  $K$  is uniquely determined by its support function. A set  $C \subset \mathbb{R}^n$  is a *convex cone* if  $\alpha_1 u_1 + \alpha_2 u_2 \in C$  for any  $u_1, u_2 \in C$  and  $\alpha_1, \alpha_2 \geq 0$ .

If  $S$  is a convex set in  $\mathbb{R}^n$ , then  $z \in S$  is an extremal point if  $z = \alpha x_1 + (1 - \alpha)x_2$  for  $x_1, x_2 \in S$  and  $\alpha \in (0, 1)$  imply  $x_1 = x_2 = z$ . We note that if  $S$  is compact and convex, then  $S$  is the convex hull of its extremal points. If  $C$  is a convex cone and  $u \in C \setminus \{o\}$ , we say that  $\sigma = \{\lambda u : \lambda \geq 0\}$  is an extremal ray if  $\alpha_1 x_1 + \alpha_2 x_2 \in \sigma$  for  $x_1, x_2 \in C$  and  $\alpha_1, \alpha_2 > 0$  imply  $x_1, x_2 \in \sigma$ . Now if  $C \neq \{o\}$  is a closed convex cone such that the origin is an extremal point of  $C$ , then  $C$  is the convex hull of its extremal rays.

The *normal cone* of a convex body  $K$  at  $z \in K$  is defined as

$$N(K, z) = \{u \in \mathbb{R}^n : \langle u, y \rangle \leq \langle u, z \rangle \text{ for all } y \in K\}$$

where  $N(K, z) = \{o\}$  if  $z \in \text{int}K$  and  $\dim N(K, z) \geq 1$  if  $z \in \partial K$ . This definition can be written also as

$$(2.2) \quad N(K, z) = \{u \in \mathbb{R}^n : h_K(u) = \langle z, u \rangle\}.$$

In particular,  $N(K, z)$  is a closed convex cone such that the origin is an extremal point, and

$$(2.3) \quad h_K(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 h_K(u_1) + \alpha_2 h_K(u_2) \text{ for } u_1, u_2 \in N(K, z) \text{ and } \alpha_1, \alpha_2 > 0.$$

A convex body  $K$  is  $C^1$ -smooth at  $p \in \partial K$  if  $N(K, p)$  is a ray, and  $\partial K$  is  $C^1$  if each  $p \in \partial K$  is a  $C^1$ -smooth point. Therefore,  $\partial K$  is  $C^1$  if and only if the restriction of  $h_K$  to any hyperplane not containing  $o$  is strictly convex, by (2.3).

We say that a convex body  $K$  is *strictly convex* if  $\partial K$  contains no segment. The *face* of  $K$  with outer normal  $u \in \mathbb{R}^n$  is defined as

$$F(K, u) = \{z \in K : h_K(u) = \langle z, u \rangle\},$$

which lies in  $\partial K$  if  $u \neq o$ . Schneider [47, Thm. 1.7.4] proves that

$$(2.4) \quad \partial h_K(u) = F(K, u).$$

Therefore,  $K$  is strictly convex if and only if  $h_K$  is  $C^1$  on  $\mathbb{R}^n \setminus \{o\}$ .

A crucial notion for this paper is the one of *surface area measure*  $S_K$  of a convex body  $K$ , which is a Borel measure on  $S^{n-1}$ , defined as follows. For any Borel set  $\omega \subset S^{n-1}$ :

$$S_K(\omega) = \mathcal{H}^{n-1}(\cup_{u \in \omega} F(K, u)) = \mathcal{H}^{n-1}(\cup_{u \in \omega} \partial h_K(u)),$$

Hence,  $S_K$  is the analogue of the Monge-Ampère measure for the restriction of  $h_K$  to  $S^{n-1}$ .

Given a convex body  $K$  containing  $o$  and  $p < 1$ , let  $S_{K,p}$  denote the  $L_p$  area measure of  $K$ ; namely,

$$(2.5) \quad dS_{K,p} = h_K^{1-p} dS_K.$$

Let  $f$  be a positive and measurable function on  $S^{n-1}$ ; we say that  $h_K$  is a solution of (1.2) in the sense of measure if

$$(2.6) \quad \int_{\omega} h_K(y)^{1-p} dS_K(y) = \int_{\omega} f(y) d\mathcal{H}^{n-1}(y)$$

for every Borel subset  $\omega$  of  $S^{n-1}$ .

In what follows we will always assume that  $f$  is bounded between two positive constants. Our first remark is that the previous definition is equivalent to the following conditions (a) and (b):

(a)  $\dim N(K, o) < n$ ; or equivalently,

$$(2.7) \quad \mathcal{H}^{n-1}(\{y \in S^{n-1} : h_K(y) = 0\}) = \mathcal{H}^{n-1}(N(K, o) \cap S^{n-1}) = 0,$$

(b) for each Borel set  $\omega \subset \{y \in S^{n-1} : h_K(y) > 0\}$ , we have

$$(2.8) \quad \int_{\omega} h_K^{1-p}(y) dS_K(y) = \int_{\omega} f(y) d\mathcal{H}^{n-1}(y).$$

Moreover, condition (b) is in turn equivalent to

(b') for each Borel set  $\omega \subset \{y \in S^{n-1} : h_K(y) > 0\}$ , we have

$$(2.9) \quad S_K(\omega) = \int_{\omega} f(y) h_K(y)^{p-1} d\mathcal{H}^{n-1}(y).$$

To prove that (b) and (b') are equivalent is a simple exercise (in which one has to take into account the fact that  $h_K$  is continuous). Indeed, both claims are in turn equivalent to the following fact: the measure  $S_K$  is absolutely continuous with respect to  $\mathcal{H}^{n-1}$  on  $S^{n-1} \setminus \{y \in S^{n-1} : h_K(y) = 0\}$ , and the Radon-Nikodym derivative of  $S_K$  with respect to  $\mathcal{H}^{n-1}$  is  $fh_K^{p-1}$ .

Let us prove the equivalence between (2.6) and (a)-(b). To this end, it will be useful the following observation: the set

$$\{x \in \mathbb{R}^n : h_K(x) = 0\}$$

is a closed convex cone. Indeed, it is the set where the non-negative, convex and 1-homogeneous function  $h_K$  attains its minimum. For convenience, we set  $\omega_0 = \{y \in S^{n-1} : h_K(y) = 0\}$ . Assume that (2.6) holds; then (b) follows immediately. If, by contradiction, (a) is false, then  $\omega_0$  has non-empty interior so that

$$0 = \int_{\omega_0} h_K(y)^{1-p} dS_K(y) = \int_{\omega_0} f(y) d\mathcal{H}^{n-1}(y) > 0,$$

i.e. a contradiction (in the last inequality we have used the fact that  $f$  is bounded from below by a positive constant). Vice versa, assume that (a) and (b) hold. Given a Borel subset  $\omega$  of  $S^{n-1}$  we may write it as the disjoint union of  $\omega' = \omega \cap \omega_0$  and  $\omega'' = \omega \setminus \omega'$ . By (a),  $\mathcal{H}^{n-1}(\omega') = 0$ , moreover  $h_K = 0$  on  $\omega'$ ; hence,

$$\begin{aligned} \int_{\omega} h_K(y)^{1-p} dS_K(y) &= \int_{\omega''} h_K(y)^{1-p} dS_K(y) \\ &= \int_{\omega''} f(y) d\mathcal{H}^{n-1}(y) \\ &= \int_{\omega} f(y) d\mathcal{H}^{n-1}(y), \end{aligned}$$

i.e. (2.6).

Our next step is to compare the solutions considered by Chou and Wang [16] with the ones introduced here. In particular, we will show that if  $h_K$  is a solution of (1.4), then it verifies conditions (a) and (b) as well (and consequently (2.6)). Note that being a solution of (1.4) in the sense of measures means that

$$(2.10) \quad S_K(\omega) = \int_{\omega} f(y) h_K(y)^{p-1} d\mathcal{H}^{n-1}(y).$$

has to hold for every Borel subset of  $S^{n-1}$ . In particular (2.9) follows (and then (b)). Moreover, as  $S_K$  is finite,  $h_K \geq 0$  and  $f$  is bounded between two positive constants, the previous relation implies that

$$\int_{S^{n-1}} h_K(y)^{p-1} d\mathcal{H}^{n-1}(y) < +\infty.$$

As  $p - 1 < 0$ , this yields that the set  $\omega_0$  where  $h_K$  vanishes on  $S^{n-1}$  has zero  $(n - 1)$ -dimensional measure. On the other hand this is the intersection of  $S^{n-1}$  with a convex cone. Hence we get condition (a).

In addition, if we now apply (2.10) to  $\omega_0$ , we get that when  $h_K$  is a solution of (1.4) then

$$(2.11) \quad S_K(N(K, o) \cap S^{n-1}) = 0.$$

Note that (2.11) implies that  $\mathcal{H}^{n-1}(X_0) = 0$ , in the notation of Theorem 1.2, because  $X_0 \subset \cup\{F(K, u) : u \in N(K, o) \cap S^{n-1}\}$  and (2.11) means, by definition,

$$\mathcal{H}^{n-1}\left(\cup_{u \in N(K, o) \cap S^{n-1}} F(K, u)\right) = 0.$$

Hence, applying Theorem 1.2 (ii) we deduce that if  $K \in \mathcal{K}_0^n$  is a solution of (1.4) for  $p < 1$  and  $f$  is bounded from below and above by positive constants, then  $\partial K$  is  $C^1$ , as it was verified by Chou and Wang [16].

### 3. SOME RESULTS ON MONGE-AMPÈRE EQUATIONS IN EUCLIDEAN SPACE

Lemma 3.1 is the tool to transfer the Monge-Ampère equation (1.2) on  $S^{n-1}$  to a Euclidean Monge-Ampère equation on  $\mathbb{R}^{n-1}$ . For  $e \in S^{n-1}$ , we consider the restriction of a solution  $h$  of (1.2) to the hyperplane tangent to  $S^{n-1}$  at  $e$ .

**Lemma 3.1.** *If  $e \in S^{n-1}$ ,  $h$  is a convex positively 1-homogeneous non-negative function on  $\mathbb{R}^n$  that is a solution of (1.2) for  $p < 1$  and positive  $f$ , and  $v(y) = h(y + e)$  holds for  $v : e^\perp \rightarrow \mathbb{R}$ , then  $v$  satisfies*

$$(3.1) \quad v^{1-p} \det(D^2 v) = g \quad \text{on } e^\perp$$

where, for  $y \in e^\perp$ , we have

$$g(y) = (1 + \|y\|^2)^{-\frac{n+p}{2}} f\left(\frac{e + y}{\sqrt{1 + \|y\|^2}}\right).$$

*Proof.* Let  $h = h_K$  for  $K \in \mathcal{K}_0^n$ , and let

$$\tilde{S} = \{u \in S^{n-1} : h_K(u) = 0\},$$

which is a possibly empty spherically convex compact set whose spherical dimension is at most  $n - 2$ , by (2.7). According to (2.9), the Monge-Ampère equation for  $h_K$  can be written in the form

$$(3.2) \quad dS_K = h_K^{p-1} f d\mathcal{H}^{n-1} \quad \text{on } S^{n-1} \setminus \tilde{S}.$$

We consider  $\pi : e^\perp \rightarrow S^{n-1}$  defined by

$$\pi(x) = (1 + \|x\|^2)^{-\frac{1}{2}}(x + e),$$



which is induced by the radial projection from the tangent hyperplane  $e + e^\perp$  to  $S^{n-1}$ . Since  $\langle \pi(x), e \rangle = (1 + \|x\|^2)^{-\frac{1}{2}}$ , the Jacobian of  $\pi$  is

$$(3.3) \quad \det D\pi(x) = (1 + \|x\|^2)^{-\frac{n}{2}}.$$

For  $x \in e^\perp$ , (2.4) and writing  $h_K$  in terms of an orthonormal basis of  $\mathbb{R}^n$  containing  $e$ , yield that  $v$  satisfies

$$\partial v(x) = \partial h_K(x + e)|_{e^\perp} = F(K, x + e)|_{e^\perp} = F(K, \pi(x))|_{e^\perp}.$$

Let  $S = \pi^{-1}(\tilde{S})$ . For a Borel set  $\omega \subset e^\perp \setminus S$ , we have

$$\begin{aligned} \mathcal{H}^{n-1}(N_v(\omega)) &= \mathcal{H}^{n-1}(\cup_{x \in \omega} \partial v(x)) \\ &= \mathcal{H}^{n-1}(\cup_{u \in \pi(\omega)} (F(K, u)|_{e^\perp})) = \int_{\pi(\omega)} \langle u, e \rangle dS_K(u) \\ &= \int_{\pi(\omega)} \langle u, e \rangle h_K^{p-1}(u) f(u) d\mathcal{H}^{n-1}(u) \\ &= \int_{\omega} (1 + \|x\|^2)^{-\frac{n-p}{2}} f(\pi(x)) v(x)^{p-1} d\mathcal{H}^{n-1}(x) \end{aligned}$$

where we used at the last step that

$$v(x) = h_K(x + e) = (1 + \|x\|^2)^{\frac{1}{2}} h_K(\pi(x)).$$

In particular,  $v$  satisfies the Monge-Ampère type differential equation

$$\det D^2 v(x) = (1 + \|x\|^2)^{-\frac{n-p}{2}} f(\pi(x)) v(x)^{p-1} \quad \text{on } e^\perp \setminus S.$$

Since  $\dim S \leq n - 2$  by (1.3),  $v$  satisfies (3.1) on  $e^\perp$ .  $\square$

Having Lemma 3.1 at hand showing the need to understand related Monge-Ampère equations in Euclidean spaces, we prove Propositions 1.3 and 3.4, and quote Caffarelli's Theorem 3.6.

*Proof of Proposition 1.3.* Up to changing coordinate system, we may assume, without loss of generality, that  $S \subset \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : x_2 = 0\}$  and the origin is contained in the relative interior of  $S$ . Therefore, up to restricting  $\Omega$ , we may also assume that  $v$  is continuous on  $\text{cl } \Omega$ , that  $\Omega = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| < s_1, \|x_2\| < s_2\}$  for some constants  $s_1, s_2 > 0$  and that  $S = \{(x_1, x_2) \in \Omega : x_2 = 0\}$ .

Let  $\alpha = \max_{\text{cl } \Omega} v$  and let us consider the convex body

$$M = \{(x_1, x_2, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R} : \|x_1\| \leq s_1, \|x_2\| \leq s_2, v(x_1, x_2) \leq y \leq \alpha\}.$$

For  $t \in (0, s_2/2]$ , let

$$\Omega_t = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| \leq s_1/2, \|x_2\| \leq t\}.$$

We estimate  $\mathcal{H}^n(N_v(\Omega_t \setminus S))$ . Let  $(x_1, x_2) \in \Omega_t \setminus S$  and let  $(z_1, z_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$  belong to  $\partial v(x_1, x_2)$ . We prove that

$$(3.4) \quad \|z_2\| \leq \frac{2\alpha}{s_2} \quad \text{and} \quad \|z_1\| \leq \frac{4\alpha}{s_1 s_2} t.$$

If  $z_2 = 0$  the first inequality in (3.4) holds true. Assume  $z_2 \neq 0$ . The vector  $(z_1, z_2, -1)$  is an exterior normal to  $M$  at  $p = (x_1, x_2, v(x_1, x_2))$ . Since

$$q_1 = \left( x_1, x_2 + \frac{s_2 z_2}{2\|z_2\|}, \alpha \right) \in M$$

(because  $\|x_2 + s_2 z_2 / (2\|z_2\|)\| \leq \|x_2\| + s_2/2 \leq s_2$ ) then  $\langle q_1 - p, (z_1, z_2, -1) \rangle \leq 0$ . This implies

$$\|z_2\| \leq \frac{2}{s_2} (\alpha - v(x_1, x_2))$$

and the first inequality in (3.4). Again, if  $z_1 = 0$  then the second inequality (3.4) holds true. Assume  $z_1 \neq 0$ . We have

$$q_2 = \left( x_1 + \frac{s_1 z_1}{2\|z_1\|}, 0, v(x_1, x_2) \right) \in M,$$

because  $\|x_1 + s_1 z_1 / (2\|z_1\|)\| \leq s_1$ ,  $(x_1 + s_1 z_1 / (2\|z_1\|), 0) \in S$  and therefore  $v(x_1, x_2) \geq 0 = v(x_1 + s_1 z_1 / (2\|z_1\|), 0)$ . The inequality  $\langle q_2 - p, (z_1, z_2, -1) \rangle \leq 0$  implies the second inequality (3.4).

The inequalities in (3.4) imply

$$(3.5) \quad \mathcal{H}^n(N_v(\Omega_t \setminus S)) \leq c t^r,$$

for a suitable constant  $c$  independent of  $t$ .

Now we estimate  $\int_{\Omega_t \setminus S} v(x)^{p-1} dx$ . The inclusion of the convex hull of  $S \times \{0\}$  and  $\{\|x_1\| \leq s_1, \|x_2\| \leq s_2, y = \alpha\}$  in  $M$  implies that  $v(x_1, x_2) \leq \frac{\alpha}{s_2} \|x_2\|$  for each  $(x_1, x_2) \in \Omega_t$  by the convexity of  $v$ . Using this estimate it is straightforward to compute that

$$(3.6) \quad \int_{\Omega_t \setminus S} v(x)^{p-1} dx \geq d t^{n+p-r-1},$$

for a suitable constant  $d$  independent on  $t$ . The inequalities (3.5) and (3.6) and the differential inequality satisfied by  $v$  imply, as  $t \rightarrow 0^+$ ,

$$c t^r \geq \mathcal{H}^n(N_v(\Omega_t \setminus S)) \geq \int_{\Omega_t \setminus S} \tau v(x)^{p-1} dx \geq \tau d t^{n+p-r-1}.$$

This inequality implies  $p \geq -n + 1 + 2r$ .  $\square$

**Example 3.2.** *Let us show that for any  $p > -n + 3$  there exists a non-negative convex solution of (1.6) in  $\Omega = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 \in [-1, 1], \|x_2\| \leq 1\}$  which vanish on the 1-dimensional space  $S = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_2 = 0\}$ .*

To prove this let

$$v(x_1, x_2) = \|x_2\| + f(\|x_2\|)g(x_1)$$

where  $f(r) = r^\alpha$ , with  $\alpha = (p + n - 1)/2$ , and  $g(x_1) = (1 + \beta x_1^2)$ , with  $\beta > 0$  sufficiently small. Note that  $\alpha > 1$  exactly when  $p > -n + 3$ .

The function  $v$  is invariant with respect to rotations around the line containing  $S$ . To compute  $\det D^2 v$  at an arbitrary point, it suffices to compute it at  $(x_1, 0, \dots, 0, r)$ ,  $r \geq 0$ . We get

$$\begin{aligned} v_{x_1 x_1} &= f(r)g''(x_1), \\ v_{x_1 x_i} &= 0 & \text{when } 1 < i < n, \\ v_{x_1 x_n} &= f'(r)g'(x_1), \\ v_{x_i x_i} &= \frac{1}{r} + \frac{f'(r)}{r}g(x_1) & \text{when } 1 < i < n, \\ v_{x_i x_j} &= 0 & \text{when } i \neq j, (i, j) \neq (1, n), (i, j) \neq (n, 1), \\ v_{x_n x_n} &= f''(r)g(x_1). \end{aligned}$$

The function  $v$  is convex if  $\beta$  is sufficiently small. Indeed, the eigenvalues of  $D^2 v$  are  $\frac{1}{r} + \frac{f'(r)}{r}g(x_1)$ , with multiplicity  $n - 2$ , and those of the matrix

$$\begin{pmatrix} f g'' & f' g' \\ f' g' & f'' g \end{pmatrix}.$$

The determinant of the latter matrix is

$$2\alpha\beta r^{2(\alpha-1)}(\alpha - 1 - (1 + \alpha)\beta x_1^2),$$

which is positive if  $\beta > 0$  is sufficiently small. Thus, all eigenvalues of  $D^2v$  are positive.

We get

$$\det D^2v = \left( f'' g f g'' - (f' g')^2 \right) \left( \frac{1}{r} + \frac{f'}{r} g \right)^{n-2}$$

which has the same order as  $r^{2\alpha-n}$  as  $r \rightarrow 0^+$ . Clearly  $v$  has order  $r$ , and  $v^{1-p} \det D^2v$  has order  $r^{2\alpha-n+1-p}$ , which is uniformly bounded from above and below for our choice of  $\alpha$ .

The next statement is a slight modification of Lemmas 3.2 and 3.3 from Trudinger and Wang [51]. Its proof closely follows that in [51] and is given here for completeness.

**Lemma 3.3.** *Let  $v$  be a convex function defined on the closure of an open bounded convex set  $\Omega \subset \mathbb{R}^n$  satisfying the Monge-Ampère equation*

$$\det D^2v = \nu$$

for a finite non-negative measure  $\nu$  on  $\Omega$ , let  $v \equiv 0$  on  $\partial\Omega$  and let  $tE \subset \Omega \subset E$  for  $t > 0$  and an origin centred ellipsoid  $E$ .

(i) *If  $z \in \Omega$  satisfies  $(z + sE) \cap \partial\Omega \neq \emptyset$  for  $s > 0$ , then*

$$|v(z)| \leq s^{1/n} c_0 \mathcal{H}^n(\Omega)^{1/n} \nu(\Omega)^{1/n}$$

for some  $c_0 > 0$  depending on  $n, t$ .

(ii) *If  $\nu(t\Omega) \geq b \nu(\Omega)$  for  $b > 0$ , then*

$$(3.7) \quad |v(0)| \geq c_1 \mathcal{H}^n(\Omega)^{1/n} \nu(\Omega)^{1/n}$$

for some  $c_1 > 0$  depending on  $n, t$  and  $b$ .

(iii) *If  $(z + sE) \cap \partial\Omega \neq \emptyset$  and  $\nu(t\Omega) \geq b \nu(\Omega)$  then*

$$(3.8) \quad \frac{|v(z)|}{|v(o)|} \leq \frac{c_1}{c_0} s^{1/n}.$$

When  $E = B^n$  the number  $s$  can be chosen as the distance of  $z$  from  $\partial\Omega$ . In the general case  $s$  has the same meaning in the metric induced by the norm whose unitary ball is  $E$ .

*Proof.* Let  $A$  be a linear transformation such that  $B^n = A^{-1}E$ , let  $\tilde{v}(x) = v(Ax)|\det A|^{-2/n}$ ,  $\tilde{\Omega} = A^{-1}\Omega$  and let  $\tilde{\nu}$  be the measure defined for each Borel set  $\omega \subset \tilde{\Omega}$  as  $\tilde{\nu}(\omega) = \nu(A\omega)/|\det A|$ . It is known that  $\tilde{v}$  solves

$$(3.9) \quad \det D^2\tilde{v} = \tilde{\nu} \quad \text{in } \tilde{\Omega}.$$

Moreover,  $tB^n \subset \tilde{\Omega} \subset B^n$ . Since  $\mathcal{H}^n(\Omega) = |\det A| \mathcal{H}^n(\tilde{\Omega})$ , we have

$$(3.10) \quad \frac{\mathcal{H}^n(\Omega)}{\omega_n} \leq |\det A| \leq \frac{\mathcal{H}^n(\Omega)}{\omega_n t^n}.$$

Let us prove Claim (i). Let  $\tilde{z} = A^{-1}z$ . Then  $(\tilde{z} + sB^n) \cap \partial\tilde{\Omega} \neq \emptyset$  and if  $d$  denotes the distance of  $\tilde{z}$  from  $\partial\tilde{\Omega}$  we have  $d \leq s$ . By choosing proper coordinates we may assume that  $\tilde{z} = (0, \dots, 0, d)$ , and that  $\tilde{\Omega} \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ . Then

$$\tilde{\Omega} \subset \tilde{\Omega} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \|(x_1, \dots, x_{n-1})\| < 2, 0 < x_n < 4\}.$$

Let  $u$  and  $w$  be convex functions such that their graphs are convex cones with vertex at  $(\tilde{z}, \tilde{v}(\tilde{z}))$  and bases  $\partial\tilde{\Omega}$  and  $\partial\tilde{\Omega}$ , respectively. Then

$$(3.11) \quad N_{\tilde{v}}(\tilde{\Omega}) \supset N_u(\tilde{\Omega}) = \partial u(\tilde{z}) \supset \partial w(\tilde{z}).$$

Since  $w$  is a convex cone over the cylinder  $\widehat{\Omega}$ , one can easily compute that  $\mathcal{H}^n(\partial w(\tilde{z})) \geq c_2 |\tilde{v}(\tilde{z})|^n/d$ , for a suitable constant  $c_2 > 0$ . This inequality, (3.9) and (3.11) imply

$$|\tilde{v}(\tilde{z})| \leq \left(\frac{d}{c_2}\right)^{1/n} \mathcal{H}^n(N_{\tilde{v}}(\tilde{\Omega}))^{1/n} = \left(\frac{d}{c_2}\right)^{1/n} \tilde{\nu}(\tilde{\Omega})^{1/n}.$$

Expressing this inequality in terms of  $v$ ,  $\Omega$  and  $\nu$  and using  $d \leq s$  and (3.10) concludes the proof of Claim (i).

Let us prove Claim (ii). There exists a unique solution  $w$  of  $\det D^2 w = \widehat{\nu}$  in  $\tilde{\Omega}$ ,  $w = 0$  in  $\partial\tilde{\Omega}$ , where  $\widehat{\nu} = \tilde{\nu}$  in  $t\tilde{\Omega}$  and  $\widehat{\nu} = 0$  elsewhere (see Theorem 2.1 in [51]). The comparison principle for Monge-Ampère equations (see Lemma 2.4 in [51]) implies  $w \geq \tilde{v}$  in  $\tilde{\Omega}$ .

Let  $z \in t\tilde{\Omega}$ . The distance  $d$  of  $z$  from  $\partial\tilde{\Omega}$  is larger than or equal to  $(1-t)t$  (here we have used the inclusion  $tB^n \subset \tilde{\Omega}$ ). If  $y \in \partial w(z)$  and  $l(x) = \langle x, y \rangle + w(z)$  then  $l(x) \leq w(x)$  for each  $x \in \tilde{\Omega}$ , by definition of subgradient. In particular, we have  $l(x) \leq 0$  for each  $x \in \partial\tilde{\Omega}$ . This implies

$$|y| \leq \frac{|w(z)|}{d} \leq \frac{\sup_{\tilde{\Omega}} |\tilde{v}|}{t(1-t)}.$$

Therefore,

$$\mathcal{H}^n(N_w(t\tilde{\Omega})) \leq \omega_n \left( \frac{\sup_{\tilde{\Omega}} |\tilde{v}|}{t(1-t)} \right)^n.$$

This inequality, the equation satisfied by  $w$  and the condition  $\nu(t\Omega) \geq b\nu(\Omega)$  imply

$$\begin{aligned} \sup_{\tilde{\Omega}} |\tilde{v}| &\geq \frac{t(1-t)}{\omega_n^{1/n}} \mathcal{H}^n(N_w(t\tilde{\Omega}))^{1/n} = \frac{t(1-t)}{\omega_n^{1/n}} \tilde{\nu}(t\tilde{\Omega})^{1/n} \\ (3.12) \quad &\geq \frac{bt(1-t)}{\omega_n^{1/n}} \tilde{\nu}(\tilde{\Omega})^{1/n}. \end{aligned}$$

We claim that

$$(3.13) \quad |\tilde{v}(o)| \geq \frac{t}{1+t} \sup_{\tilde{\Omega}} |\tilde{v}|.$$

Indeed, let  $z \in \tilde{\Omega}$  be such that  $\tilde{v}(z) = \inf_{\tilde{\Omega}} \tilde{v}$ . We may clearly assume  $z \neq 0$ , since otherwise there is nothing to prove. By choosing proper coordinates we may assume  $z = (z_1, 0, \dots, 0)$  for some  $z_1 > 0$ . Let  $l$  be the linear function defined on the line through  $o$  and  $z$  and such that  $l(o) = \tilde{v}(o)$  and  $l(z) = \tilde{v}(z)$ . It is  $l(s, 0, \dots, 0) = \tilde{v}(o) + s(\inf_{\tilde{\Omega}} \tilde{v} - \tilde{v}(o))/z_1$ . Since  $\tilde{v}$  is convex,

$$l(s, 0, \dots, 0) \leq \tilde{v}(s, 0, \dots, 0)$$

for each  $s \notin [0, z_1]$  such that  $(s, 0, \dots, 0) \in \tilde{\Omega}$ . When  $s = -t$  we obtain  $l(-t, 0, \dots, 0) \leq \tilde{v}(-t, 0, \dots, 0) \leq 0$ . The inequality  $l(-t, 0, \dots, 0) \leq 0$  and the inclusion  $\tilde{\Omega} \subset B^n$  imply (3.13).

The proof of Claim (ii) is concluded by combining (3.12) and (3.13) and expressing the obtained inequality in terms of  $v$ ,  $\Omega$  and  $\nu$ .

Claim (iii) is a consequence of the first two claims.  $\square$

The proof of Claim (ii) in Theorem 1.2 is based on the following proposition, which is related to a step in the proof of Theorem E (a) in [16], however our proof is substantially different from that in [16].

**Proposition 3.4.** *Let  $v$  be a non-negative convex function defined on the closure of an open convex set  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , such that  $S = \{x \in \Omega : v(x) = 0\}$  is non-empty*

and compact, and  $v$  is locally strictly convex on  $\Omega \setminus S$ . Let  $\psi : (0, \infty) \rightarrow [0, \infty)$  be monotone decreasing and not identically zero; assume that  $\tau_2 > \tau_1 > 0$  and  $v$  satisfy

$$(3.14) \quad \tau_1 \psi(v) \leq \det D^2 v \leq \tau_2 \psi(v)$$

in the sense of measure on  $\Omega \setminus S$ . If  $\dim S \leq n-1$  and  $\mu_v(S) = 0$  for the associated Monge-Ampère measure  $\mu_v$ , then  $S$  is a point.

Note that (3.14) means that for each Borel set  $\omega \subset \Omega \setminus S$  we have

$$\tau_1 \int_{\omega} \psi(v(x)) dx \leq \mu_v(\omega) \leq \tau_2 \int_{\omega} \psi(v(x)) dx,$$

where  $\mu_v$  has been defined in (2.1).

*Proof.* We assume, arguing by contradiction that  $S$  is not a point. Choose coordinates so that  $o$  is the centre of mass of  $S$ . Let  $L = \text{lin } S$ . By assumption

$$(3.15) \quad 1 \leq \dim L \leq n-1.$$

Let  $e = (o, 1) \in \mathbb{R}^n \times \mathbb{R}$ . We may assume that  $\Omega$  is bounded, after possibly substituting it with a bounded open neighbourhood of  $S$ . We start by illustrating the idea of the proof.

*Sketch of the proof.* For any small  $\varepsilon > 0$ , we construct an affine function  $l_\varepsilon$  such that  $l_\varepsilon(x) = \varepsilon$  for  $x \in L$ , and the convex set  $\Omega_\varepsilon = \{v < l_\varepsilon\}$  is well-balanced; namely, there exists an ellipsoid  $E_\varepsilon$  centred at the origin such that  $(1/(8n^3))E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$  (see (3.19)). This is the longest part of the argument, and the main idea to construct  $l_\varepsilon$  is that the graph of  $l_\varepsilon$  cuts off the smallest volume cap from the graph of  $v$  among the hyperplanes in  $\mathbb{R}^{n+1}$  containing  $L + \varepsilon e$ . Subsequently, we apply Lemma 3.3 to  $\Omega_\varepsilon$  and to the function  $v - l_\varepsilon$  in the standard way to reach a contradiction. We show that one can choose  $z \in S$  so that the corresponding parameter  $s$ , as defined in Lemma 3.3, tends to 0 as  $\varepsilon$  tends to 0. (Equivalently,  $S$  contains points whose distance from  $\partial\Omega_\varepsilon$ , the one induced by the norm whose unit ball is  $E_\varepsilon$ , tends to 0 as  $\varepsilon$  tends to 0.) This contradicts (3.8), since  $|v(z) - l_\varepsilon(z)|/|v(o) - l_\varepsilon(o)| = \varepsilon/\varepsilon = 1$ .

We divide the proof into four steps.

*Step 1. Definition of  $l_\varepsilon$  and of  $\Omega_\varepsilon$ .*

Let  $\varepsilon_0 = \min_{\partial\Omega} v > 0$  and let us consider the  $(n+1)$ -dimensional convex body

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : v(x) \leq y \leq \varepsilon_0\}.$$

For  $\varepsilon \in (0, \varepsilon_0)$  define  $H_\varepsilon$  to be a hyperplane in  $\mathbb{R}^{n+1}$

- (i) containing  $L + \varepsilon e = \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} : x \in L\}$  and
- (ii) cutting off the minimal volume from  $M$  (on the side containing the origin) under condition (i).

Let  $r > 0$ . We claim that there exists  $\varepsilon_1 = \varepsilon_1(r)$  so that  $H_\varepsilon$  is the graph of an affine function  $l_\varepsilon$  for each  $\varepsilon \in (0, \varepsilon_1)$ , and, setting

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : v(x) < l_\varepsilon(x)\},$$

we have

$$(3.16) \quad \text{cl}\Omega_\varepsilon \subset \Omega, \quad S \subset \Omega_\varepsilon \quad \text{and} \quad \Omega_\varepsilon \cap L \subset (1+r)S.$$

Let  $F = \{(x, y) \in M : y = \varepsilon_0\}$  be the upper face of  $M$  and let  $\mathcal{H}$  be the collection of hyperplanes in  $\mathbb{R}^{n+1}$  which intersect both  $F$  and  $\{(x, y) \in M : y \leq \varepsilon_0/2\}$ . Since  $\Omega$  is bounded and  $v$  is locally strictly convex on  $\Omega \setminus S$ , every hyperplane in  $\mathcal{H}$  is not a supporting hyperplane to  $M$ . Therefore, by compactness, there exists a constant  $\varrho_0 > 0$  such that for every  $H \in \mathcal{H}$  both components of  $M \setminus H$  are of volume at least  $\varrho_0$ . We choose  $\varepsilon_1 \in (0, \varepsilon_0/2)$  such that the volume of the cap  $\{(x, y) \in M : y \leq \varepsilon_1\}$  is less than  $\varrho_0$ . This choice implies that the minimum value of the problem which

defines  $H_\varepsilon$  is less than  $\varrho_0$ . Therefore, a minimizer  $H_\varepsilon$  does not belong to  $\mathcal{H}$ . Since  $H_\varepsilon \cap \{(x, y) \in M : y \leq \varepsilon_0/2\} \neq \emptyset$ , we have  $H_\varepsilon \cap F = \emptyset$ . In particular,  $H_\varepsilon$  is the graph of a affine function defined on  $\mathbb{R}^n$  and  $\text{cl}\Omega_\varepsilon \subset \Omega$ .

The inclusion  $S \subset \Omega_\varepsilon$  holds because  $v(x) = 0$  and  $l_\varepsilon(x) = \varepsilon$  for any  $x \in S$ .

The origin  $o$ , being the centre of mass of  $S$ , belongs to the relative interior of  $S$ . Since  $\dim S > 0$ , the relative boundary of  $(1+r)S$  does not intersect  $S$ . This implies  $\inf_{\text{relbd}(1+r)S} v > 0$ . Thus, if  $\varepsilon_1$  satisfies

$$\varepsilon_1 < \inf_{\text{relbd}(1+r)S} v$$

in addition to the inequalities specified above then  $v(x) > \varepsilon$  and  $l_\varepsilon(x) = \varepsilon$  for any  $x \in \text{relbd}(1+r)S$  ( $l_\varepsilon(x) = \varepsilon$  is a consequence of  $(1+r)S \subset L$ ). This implies  $\Omega_\varepsilon \cap L \subset (1+r)S$ .

In the rest of the proof we may assume  $\varepsilon_1 < \varepsilon_1(1)$  so that

$$(3.17) \quad \Omega_\varepsilon \cap L \subset 2S.$$

*Step 2. The centre of mass of  $\Omega_\varepsilon$  is contained in  $L$ .*

To prove this claim we have to prove that for each  $w \in L^\perp \cap \mathbb{R}^n$  we have

$$(3.18) \quad \int_{\Omega_\varepsilon} \langle x, w \rangle dx = 0.$$

Indeed, for  $t \in \mathbb{R}$  with  $|t|$  small, let

$$F(t) = \int_{\{x \in \Omega : l_\varepsilon(x) + t\langle x, w \rangle - v(x) > 0\}} (l_\varepsilon(x) + t\langle x, w \rangle - v(x)) dx$$

be the volume cut off by the hyperplane in  $\mathbb{R}^{n+1}$  that is the graph of  $x \mapsto l_\varepsilon(x) + t\langle x, w \rangle$  from  $M$ . By definition of  $H_\varepsilon$  and  $l_\varepsilon$ ,  $F$  has a local minimum at  $t = 0$ . We have

$$\begin{aligned} \frac{F(t) - F(0)}{t} &= \int_{\{x \in \Omega : l_\varepsilon(x) - v(x) > 0\}} \langle x, w \rangle dx \\ &+ \int_{\Omega} \left( \frac{l_\varepsilon(x) - v(x)}{t} + \langle x, w \rangle \right) \left( 1_{\{x : l_\varepsilon(x) + t\langle x, w \rangle - v(x) > 0\}} - 1_{\{x : l_\varepsilon(x) - v(x) > 0\}} \right) dx. \end{aligned}$$

The set where  $1_{\{x : l_\varepsilon(x) + t\langle x, w \rangle - v(x) > 0\}} - 1_{\{x : l_\varepsilon(x) - v(x) > 0\}}$  differs from 0 is contained in

$$A_t = \{x \in \Omega : |l_\varepsilon(x) - v(x)| < |t\langle x, w \rangle|\}$$

and there exists  $c$  independent on  $t$  such that  $\mathcal{H}^n(A_t) < ct$  and  $\sup_{A_t} |l_\varepsilon(x) - v(x)| < ct$ . As  $F$  has a local minimum at  $t = 0$ , we have

$$0 = \frac{dF}{dt}(0) = \int_{\Omega_\varepsilon} \langle x, w \rangle dx,$$

which proves (3.18).

*Step 3. For any  $\varepsilon \in (0, \varepsilon_1)$  there exists an ellipsoid  $E_\varepsilon$  centred at the origin such that*

$$(3.19) \quad \frac{1}{8n^3} E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon.$$

Lemma 2.3.3 in [47] proves that any  $k$ -dimensional convex body contains its reflection, with respect to its centre of mass, scaled, with respect to the same centre of mass, by  $1/k$ . From the fact that the centre of mass of  $\Omega_\varepsilon$  belongs to  $L$  we deduce that

$$(3.20) \quad -(\Omega_\varepsilon | L^\perp) \subset n(\Omega_\varepsilon | L^\perp).$$

According to Loewner's or John's theorems, there exists an ellipsoid  $\tilde{E}$  centred at the origin and  $z_1 \in \Omega_\varepsilon$  such that

$$z_1 + \frac{1}{n} \tilde{E} \subset \Omega_\varepsilon \subset z_1 + \tilde{E}.$$

It follows from (3.20) that there exists  $z_2 \in \Omega_\varepsilon$  such that  $z_2|L^\perp = \frac{-1}{n} z_1|L^\perp$ . In particular,  $y_1 = \frac{1}{n+1} z_1 + \frac{n}{n+1} z_2 \in \Omega_\varepsilon$  verifies  $y_1|L^\perp = o$ , or in other words,  $y_1 \in L \cap \Omega_\varepsilon$ . In addition,

$$y_1 + \frac{1}{2n^2} \tilde{E} \subset \frac{1}{n+1} \left( z_1 + \frac{1}{n} \tilde{E} \right) + \frac{n}{n+1} z_2 \subset \Omega_\varepsilon.$$

Let  $m = \dim L \leq n-1$ . Since  $y_1 \in L \cap \Omega_\varepsilon$  and (3.17) imply  $\frac{1}{2} y_1 \in S$ , and since the origin is the centroid of  $S$ , we deduce that  $y_2 = \frac{-1}{2m} y_1 \in S$ . As  $2m+1 < 2n$ , we have

$$\frac{1}{4n^3} \tilde{E} \subset \frac{1}{2m+1} \left( y_1 + \frac{1}{2n^2} \tilde{E} \right) + \frac{2m}{2m+1} y_2 \subset \Omega_\varepsilon.$$

As  $\Omega_\varepsilon \subset 2\tilde{E}$  follows from  $o \in z_1 + \tilde{E}$ , we may choose  $E_\varepsilon = 2\tilde{E}$ , proving (3.19).

*Step 4. Application of Lemma 3.3 to  $v - l_\varepsilon$  and  $\Omega_\varepsilon$  and contradiction.*

We observe that

$$(3.21) \quad v(x) - l_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \partial\Omega_\varepsilon \\ -\varepsilon & \text{if } x \in S. \end{cases}$$

Let  $\nu$  denote the Monge-Ampère measure  $\mu_{(v-l_\varepsilon)}$  restricted to  $\Omega_\varepsilon$ . If  $\Omega_0$  is an open set such that  $\Omega_\varepsilon \subset \Omega_0 \subset \text{cl } \Omega_0 \subset \Omega$ , then the set  $N_v(\Omega_0)$  is bounded and this implies

$$\nu(\Omega_\varepsilon) = \mathcal{H}^n(N_{(v-l_\varepsilon)}(\Omega_\varepsilon)) \leq \mathcal{H}^n(N_v(\Omega_0)) < \infty.$$

Let  $t = 1/(8n^3)$ . Formula (3.19) yields  $tE_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$ . Let us prove that

$$(3.22) \quad \nu(t\Omega_\varepsilon) \geq b\nu(\Omega_\varepsilon) \quad \text{for } b = \tau_1 t^n / \tau_2.$$

The function  $v$  is convex and attains its minimum at  $o$ , thus  $v(x) \geq v(tx)$  for any  $x \in \Omega_\varepsilon$ . By this fact, the monotonicity of  $\psi$ , (3.14) and the assumptions on  $S$ , we deduce that

$$\begin{aligned} \nu(t\Omega_\varepsilon) &= \nu(t(\Omega_\varepsilon \setminus S)) \geq \tau_1 \int_{t(\Omega_\varepsilon \setminus S)} \psi(v(x)) dx \\ &= \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(tz)) dz \\ &\geq \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(z)) dz \\ &\geq \frac{\tau_1 t^n}{\tau_2} \nu(\Omega_\varepsilon \setminus S) = \frac{\tau_1 t^n}{\tau_2} \nu(\Omega_\varepsilon) \end{aligned}$$

proving (3.22).

Let  $z \in \text{relbd } S$ . We claim that when  $\varepsilon \in (0, \varepsilon_1(r))$  then  $(z + rE_\varepsilon) \cap \partial\Omega_\varepsilon \neq \emptyset$ . This is a consequence of the second and third inclusion in (3.16). Indeed, since  $o \in S \subset \Omega_\varepsilon \subset E_\varepsilon$ , there exists  $q_\varepsilon > 0$  such that  $(1 + q_\varepsilon)z \in \partial E_\varepsilon$ . The set  $z + rE_\varepsilon$  contains the segment  $[z, z + r(1 + q_\varepsilon)z]$ . Since  $q_\varepsilon > 0$ , that segment contains the segment  $[z, (1 + r)z]$ . The second and third inclusion in (3.16) imply  $[z, (1 + r)z] \cap \partial\Omega_\varepsilon \neq \emptyset$ . This proves the claim.

Lemma 3.3 applies to this situation with  $s = r$ . Since  $v(z) - l_\varepsilon(z) = v(o) - l_\varepsilon(o) = -\varepsilon$  (see (3.21)), (3.8) yields

$$1 = \frac{|v(z) - l_\varepsilon(z)|}{|v(o) - l_\varepsilon(o)|} \leq \frac{c_1}{c_0} r^{1/n}.$$

Since  $r$  can be any positive number, we have reached a contradiction.  $\square$

We will actually use the following consequence of Proposition 3.4.

**Corollary 3.5.** *Let  $\tau_2 > \tau_1 > 0$ , and let  $g$  be a function defined on an open convex set  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , such that  $\tau_2 > g(x) > \tau_1$  for  $x \in \Omega$ . For  $p < 1$ , let  $v$  be a non-negative convex solution of*

$$v^{1-p} \det D^2 v = g \quad \text{in } \Omega.$$

*If  $S = \{x \in \Omega : v(x) = 0\}$  is non-empty, compact and  $\mu_v(S) = 0$ , and  $v$  is locally strictly convex on  $\Omega \setminus S$ , then  $S$  is a point.*

*Proof.* All we have to check that  $\dim S \leq n - 1$ . It follows from the fact that the left-hand side of the differential equation is zero on  $S$ , while the right-hand side is positive.  $\square$

The following result by L. Caffarelli (see Theorem 1 and Corollary 1 in [7]) is the key in handling the regularity and strict convexity of the part of the boundary of a convex body  $K$  where the support function at some normal vector is positive.

**Theorem 3.6** (Caffarelli). *Let  $\lambda_2 > \lambda_1 > 0$ , and let  $v$  be a convex function on an open convex set  $\Omega \subset \mathbb{R}^n$  such that*

$$(3.23) \quad \lambda_1 \leq \det D^2 v \leq \lambda_2$$

*in the sense of measure.*

- (i) *If  $v$  is non-negative and  $S = \{x \in \Omega : v(x) = 0\}$  is not a point, then  $S$  has no extremal point in  $\Omega$ .*
- (ii) *If  $v$  is strictly convex, then  $v$  is  $C^1$ .*

We recall that (3.23) is equivalent to saying that for each Borel set  $\omega \subset \Omega$  we have

$$\lambda_1 \mathcal{H}^n(\omega) \leq \mu_v(\omega) \leq \lambda_2 \mathcal{H}^n(\omega),$$

where  $\mu_v$  has been defined in (2.1).

#### 4. PROOF OF THEOREM 1.1

The next lemma provides a tool for the proof of Theorem 1.1 (iii). The same result is also proved in Chou and Wang [16]; we present a short argument for the sake of completeness.

**Lemma 4.1.** *For  $n \geq 2$  and  $p \leq 2 - n$ , if  $K \in \mathcal{K}_0^n$  and there exists  $c > 0$  such that  $S_{K,p}(\omega) \geq c \mathcal{H}^{n-1}(\omega)$  for any Borel set  $\omega \subset S^{n-1}$ , then  $o \in \text{int } K$ .*

*Proof.* We suppose that  $o \in \partial K$  and seek a contradiction. We choose  $e \in N(K, o) \cap S^{n-1}$  such that  $\{\lambda e : \lambda \geq 0\}$  is an extremal ray of  $N(K, o)$ . Let  $H^+$  be a closed half space containing  $\mathbb{R}e$  on the boundary such that  $N(K, o) \cap \text{int } H^+ = \emptyset$ . Let

$$V_0 = S^{n-1} \cap (e + B^n) \cap \text{int } H^+.$$

It follows by the condition on  $S_{K,p}$  that

$$(4.1) \quad c \int_{V_0} h_K(u)^{p-1} d\mathcal{H}^{n-1} \leq \int_{V_0} h_K(u)^{p-1} dS_{K,p} = S_K(V_0) < \infty.$$

However, since  $h_K$  is convex and  $h_K(e) = 0$ , there exists  $c_0 > 0$  such that

$$h_K(x) \leq c_0 \|x - e\| \quad \text{for } x \in e + B^n.$$

We observe that the radial projection of  $V_0$  onto the tangent hyperplane  $e + e^\perp$  to  $S^{n-1}$  at  $e$  is  $e + V'_0$  for

$$V'_0 = e^\perp \cap (\sqrt{3} B^n) \cap \text{int } H^+.$$



If  $y \in V'_0$ , then  $u = (e + y)/\|e + y\|$  verifies  $\|u - e\| \geq \|y\|/2$ . It follows that

$$\begin{aligned} \int_{V_0} h_K(u)^{p-1} d\mathcal{H}^{n-1} &\geq c_0^{p-1} \int_{V_0} \|u - e\|^{p-1} d\mathcal{H}^{n-1}(u) \\ &\geq \frac{c_0^{p-1}}{2} \int_{V'_0} \frac{\|y\|^{p-1}}{(1 + \|y\|^2)^{n/2}} d\mathcal{H}^{n-1}(y) \\ &\geq \frac{c_0^{p-1}}{2^{n+1}} \int_{V'_0} \|y\|^{p-1} d\mathcal{H}^{n-1}(y) = \infty \end{aligned}$$

as  $p \leq 2 - n$ . This contradicts (4.1), and hence verifies the lemma.  $\square$

*Proof of Theorem 1.1. Claim (i).* For  $u_0 \in S^{n-1} \setminus N(K, o)$ , we choose a spherically convex open neighbourhood  $\Omega_0$  of  $u_0$  on  $S^{n-1}$  such that for any  $u \in \text{cl } \Omega_0$ , we have  $\langle u, u_0 \rangle > 0$  and  $u \notin N(K, o)$ . Let  $\Omega \subset u_0^\perp$  be defined in a way such that  $u_0 + \Omega$  is the radial image of  $\Omega_0$  into  $u_0 + u_0^\perp$ , and let  $v$  be the function on  $\Omega$  defined as in Lemma 3.1 with  $h = h_K$ . Since  $h_K$  is positive and continuous on  $\text{cl } \Omega$ , we deduce from Lemma 3.1 that there exist  $\lambda_2 > \lambda_1 > 0$  depending on  $K$ ,  $u_0$  and  $\Omega_0$  such that

$$(4.2) \quad \lambda_1 \leq \det D^2 v \leq \lambda_2$$

on  $\Omega$ .

First we claim that

$$(4.3) \quad \text{if } z \in \partial K \text{ and } N(K, z) \not\subset N(K, o), \text{ then } z \text{ is a } C^1\text{-smooth point.}$$

We suppose that  $\dim N(K, z) \geq 2$ , and seek a contradiction. Since  $N(K, z)$  is a closed convex cone such that  $o$  is an extremal point, the property  $N(K, z) \not\subset N(K, o)$  yields an  $e \in (N(K, z) \cap S^{n-1}) \setminus N(K, o)$  generating an extremal ray of  $N(K, z)$ . We apply the construction above for  $u_0 = e$ . The convexity of  $h_K$  and (2.2) imply  $h_K(x) \geq \langle z, x \rangle$  for  $x \in \mathbb{R}^n$ , with equality if and only if  $x \in N(K, z)$ . We define  $S \subset \Omega$  by  $S + e = N(K, z) \cap (\Omega + e)$  and hence  $o$  is an extremal point of  $S$ . It follows that the function  $\tilde{v}$  defined by  $\tilde{v}(y) = v(y) - \langle z, y + e \rangle$  is non-negative on  $\Omega$ , satisfies (4.2), and

$$S = \{y \in \Omega : \tilde{v}(y) = 0\}.$$

These properties contradict Caffarelli's Theorem 3.6 (i) as  $o$  is an extremal point of  $S$ , and in turn we conclude (4.3).

Next we show that

$$(4.4) \quad h_K \text{ is differentiable at any } u_0 \in S^{n-1} \setminus N(K, o).$$

We apply again the construction above for  $u_0$ . If  $u \in \Omega_0$  and  $z \in F(K, u)$  clearly  $K$  is  $C^1$ -smooth at  $z$  (i.e.  $N(K, z)$  is a ray) by (4.3). Therefore, by (2.3),  $v$  is strictly convex on  $\Omega$  and Caffarelli's Theorem 3.6 (ii) yields that  $v$  is  $C^1$  on  $\Omega$ . In turn, we conclude (4.4).

In addition,  $F(K, u)$  is a unique  $C^1$ -smooth point for  $u \in \Omega_0$  (see (2.4)), yielding that  $\Omega_* = \cup\{F(K, u) : u \in \Omega_0\}$  is an open subset of  $\partial K$ . Therefore  $\Omega_* \subset X$ , any point of  $\Omega_*$  is  $C^1$ -smooth (by (2.3)) and  $\Omega_*$  contains no segment (by (2.4)), completing the proof of Claim (i).

*Claim (ii).* We suppose that  $o \in \partial K$  is  $C^1$ -smooth, and there exists  $z \in \partial K$  such that  $K$  is not  $C^1$ -smooth at  $z$ . Claim (i) yields that  $z \in X_0$ , and hence  $N(K, z) \subset N(K, o)$ , which is a contradiction, verifying Claim (ii).

*Claim (iii).* This is a consequence of Lemma 4.1 and Claim (i).

*Claim (iv).* This is a consequence of Lemma 3.1, Claim (i) and Caffarelli [8].  $\square$

**Example 4.2.** If  $n \geq 2$  and  $p \in (-n+2, 1)$ , then there exists  $K \in \mathcal{K}_0^n$  with  $C^1$  boundary such that  $o$  lies in the relative interior of a facet of  $\partial K$  and  $dS_{K,p} = f d\mathcal{H}^{n-1}$  for a strictly positive continuous  $f : S^{n-1} \rightarrow \mathbb{R}$ .

Let  $q = (p+n-1)/(p+n-2)$ . We have  $q > 1$ . Let

$$g(r) = \begin{cases} (r-1)^q & \text{when } r \geq 1; \\ 0 & \text{when } r \in [0, 1); \end{cases}$$

and  $\bar{g}(x_1, \dots, x_{n-1}) = g(\|(x_1, \dots, x_{n-1})\|)$ . Let  $K \in \mathcal{K}_0^n$  be such that  $K \cap \{x : x_n \leq 1\} = \{x : 1 \geq x_n \geq \bar{g}(x_1, \dots, x_{n-1})\}$  and  $\partial K \cap \{x : x_n > 0\}$  is a  $C^2$  surface with Gauss curvature positive at every point. Clearly  $K \cap \{x : x_n = 0\}$  is a  $(n-1)$ -dimensional face of  $K$  which contains  $o$  in its relative interior and has unit outer normal  $(0, \dots, 0, -1)$ .

To prove that  $dS_{K,p} = f d\mathcal{H}^{n-1}$  for a positive continuous  $f : S^{n-1} \rightarrow \mathbb{R}$ , it suffices to prove that there is a neighbourhood of the South pole where  $dS_{K,p}/d\mathcal{H}^{n-1}$  is continuous and bounded from above and below by positive constants. Let  $h$  be the support function of  $K$  and, for  $y \in \mathbb{R}^{n-1}$ , let  $v(y) = h(y, -1)$  be the restriction of  $h$  to the hyperplane tangent to  $S^{n-1}$  at the South pole. It suffices to prove that in a neighbourhood  $U$  of  $o$ ,  $v$  satisfies the equation  $v^{1-p} \det D^2 v = G$  with a function  $G$  which is bounded from above and below by positive constants.

If  $y \in U \setminus \{o\}$  we have

$$(4.5) \quad v(y) = h(y, -1) = \langle (x', \bar{g}(x')), (y, -1) \rangle \quad \text{where} \quad D\bar{g}(x') = y.$$

If  $U$  is sufficiently small then  $v(y)$  depends only on  $\|y\|$ . Let  $y = (z, 0, \dots, 0)$ , with  $z > 0$  small and let  $r = 1 + (z/q)^{1/(q-1)}$ . We have

$$D\bar{g}(r, 0, \dots, 0) = (z, 0, \dots, 0)$$

and (4.5) gives

$$\begin{aligned} v(z, 0, \dots, 0) &= r q (r-1)^{q-1} - (r-1)^q \\ &= z + \frac{q-1}{q^{n-1+p}} z^{n-1+p}. \end{aligned}$$

(Note that  $n-1+p > 1$ .) Clearly  $v(0, \dots, 0) = h(0, \dots, 0, -1) = 0$ . When  $z > 0$  we have

$$\begin{aligned} v_{y_1 y_1} &= \frac{q-1}{q^{n-1+p}} (n-1-p)(n-2-p) z^{n-3+p} \\ v_{y_i y_i} &= \frac{1}{z} + \frac{q-1}{q^{n-1+p}} (n-1-p) z^{n-3+p} \quad \text{when } i \neq 1 \\ v_{y_i y_j} &= 0 \quad \text{when } i \neq j, \end{aligned}$$

and, as  $z \rightarrow 0^+$

$$v(z, 0, \dots, 0)^{1-p} \det D^2 v(z, 0, \dots, 0) = c + o(1),$$

for a suitable constant  $c > 0$ . This implies the existence of a function  $G$  positive and continuous on  $U$  such that

$$\mathcal{H}^{n-1}(N_v(\omega \cap \{v > 0\})) = \int_{\omega \cap \{v > 0\}} n G(y) v(y)^{p-1} dy.$$

for any Borel set  $\omega \subset U$ . To conclude the proof that  $v$  is a solution in the sense of Alexandrov of  $v^{1-p} \det D^2 v = G$  in  $U$  it remains to prove that  $\mathcal{H}^{n-1}(\{y \in U : v(y) = 0\}) = 0$ , but this is obvious since  $\{y \in U : v(y) = 0\} = \{o\}$ .

We remark that  $h$  is not a solution of (1.4) because (2.11) fails.

## 5. PROOFS OF THEOREM 1.2 AND COROLLARY 1.4

*Proof of Theorem 1.2.* We may assume that  $o \in \partial K$  since otherwise  $\partial K$  is  $C^1$  by Theorem 1.1. Let  $e \in N(K, o) \cap S^{n-1}$  be such that  $\langle u, e \rangle > 0$  for any  $u \in N(K, o) \cap S^{n-1}$ . Let  $v$  be defined on  $\Omega = e^\perp$  as in Lemma 3.1 with  $h = h_K$  and let  $S = \{x \in e^\perp : v(x) = 0\}$ . We have

$$(5.1) \quad S + e = N(K, o) \cap (e^\perp + e),$$

by (2.2). If  $K$  is not  $C^1$ -smooth at  $o$  then  $\dim S \geq 1$  and, by Proposition 1.3,  $p \geq n - 4$  (note that here the dimension of the ambient space is  $n - 1$ ). This proves Theorem 1.2 (i).

To prove Theorem 1.2 (ii) we observe that

$$N_{h_K}(e + S) = \bigcup_{u \in N(K, o)} F(K, u) = X_0,$$

where  $X_0$  is defined as in Theorem 1.1 (i). The equality on the left in this formula follows by (2.4) and the equality on the right follows by Theorem 1.1 (i). Thus,

$$N_v(S) = X_0|e^\perp,$$

and if  $\mathcal{H}^{n-1}(X_0) = 0$  then  $\mu_v(S) = 0$ . We observe that  $S$  is compact, by (5.1), that  $v$  is locally strictly convex, by Theorem 1.1 (i), and that  $\dim S \leq n - 2$ , by (1.3). Hence, Theorem 1.2 (ii) follows by Corollary 3.5 and (5.1).  $\square$

*Proof of Corollary 1.4.* Claim (i) is an immediate consequence of (2.2), Proposition 1.3 and Lemma 3.1. This claim implies that when  $n = 4$  or  $n = 5$  and  $\partial K$  is not  $C^1$  then  $\dim N(K, o) = 2$ . In this case  $N(K, o) \cap S^{n-1}$  is a closed arc: let  $e_1$  and  $e_2$  be its endpoints. If  $u \in N(K, o) \cap S^{n-1}$ ,  $u \neq e_1$ ,  $u \neq e_2$ , then  $F(K, u)$  is contained in the intersection of the two supporting hyperplanes  $\{x \in \mathbb{R}^n : \langle x, e_i \rangle = h_K(e_i)\}$ ,  $i = 1, 2$ . Thus,

$$\mathcal{H}^{n-1}\left(\bigcup\{F(K, u) : u \in N(K, o) \cap S^{n-1}, u \neq e_1, u \neq e_2\}\right) = 0.$$

Therefore  $\dim F(K, e_1) = n - 1$  or  $\dim F(K, e_2) = n - 1$ , because otherwise

$$\bigcup\{F(K, u) : u \in N(K, o) \cap S^{n-1}\},$$

which coincides with  $X_0$  by Theorem 1.1 (i), has  $(n - 1)$ -dimensional Hausdorff measure equal to zero and  $\partial K$  is  $C^1$  by Theorem 1.2 (ii).  $\square$

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